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***Abstract*—In this paper we investigate the range resolution of transmit waveform designs matched to extended targets. We specifically look at eigenwaveform design which is also known as SNR-based illumination waveform design. To that end, we evaluate some ambiguity functions of radar systems employing eigenwaveforms. We consider some example targets and plot the corresponding ambiguity functions. Unlike traditional waveforms whose responses totally dictate the shape of the ambiguity function, both matched illumination waveform and extended target response contribute to the shape of the ambiguity function. In other words, range and Doppler resolutions are not just functions of the transmit waveform but of the target response itself which makes for interesting ambiguity functions. Moreover, we also evaluate the detection probability of eigenwaveforms matched to extended targets and show the performance improvement over wideband pulsed waveform designs.**

***Index Terms*—waveform design, range resolution, ambiguity function, eigenwaveform**

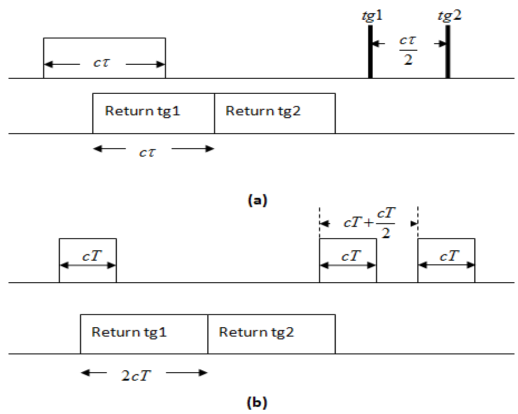

Fig. 1. (a) Range separation for point targets. (b) Range separation for extended targets

I. Introduction

For a traditional radar system where targets of interest are very far in range, a good and common model is to assume that the targets are point targets. In additive white Gaussian channel, the received waveform is therefore the scaled transmit waveform plus noise. The probability of detection is then a function of the received waveform energy regardless of its shape [1]. Moreover, the ambiguity function is dictated by the shape of the transmit waveform [2]. Waveform designs in consideration of the ambiguity function for point target model has a rich literature [2, 3] and these waveforms have well known properties in terms of range resolution, Doppler resolution, and probability of detection. Our interest however is extended targets, i.e targets that have certain impulse responses (i.e. have finite time support) and therefore the returns do not just depend on the transmit waveform but rather depend on both the transmit signal and the target's response via their convolution. Given constraint in energy, it is shown in [4] that the eigenwaveform is the waveform that maximizes the signal-to-noise ratio in additive white or colored noise. In signal-dependent interference however, the energy spectral density (ESD) of the optimal waveform is derived in [5]. In [4], the emphasis was to derive optimum waveforms for extended targets while in [5], the emphasis is to apply the optimum waveforms in a target recognition or classification using cognitive radar. For extended target, the range resolution, ambiguity function and probability of detection are clearly affected by the target's impulse response. In this paper, we investigate and evaluate the range resolution, ambiguity function, and probability of detection of a radar that employs eigenwaveform.

In Section II, we derive the range resolution for extended targets and illustrate it with an example. In Section III, we derive the ambiguity function properties for extended target illuminated by the eigenwaveform. We consider example targets and plot the corresponding ambiguity functions. Unlike traditional waveforms which totally dictate the ambiguity function, both matched illumination waveform and extended target response contribute to the shape of the ambiguity function. This fact makes for interesting ambiguity functions. In Section IV, we also evaluate the detection probability of eigenwaveform matched to an extended target. We consider an example target, its corresponding eigenwaveform and show the performance improvement over a traditional pulsed waveform design. In Section V, we present our conclusions.

II. Range Resolution for Extended Target Illuminated by an Eigenwaveform

For point targets, range resolution is dictated by the transmit waveform. It is defined as the minimum separation needed so that the return pulses do not overlap each other, i.e. it is the

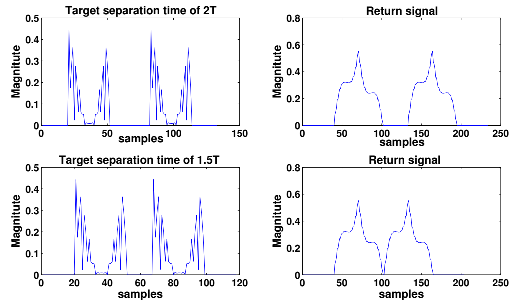

Fig. 2. Top panel: Target returns with 2T separation. Bottom panel: Target returns with 1.5T separation.

range separation required such that two point targets can be resolved. This is easily illustrated in Fig. 1a. Let τ be the time duration of the transmit waveform and c be the velocity of propagation. For the return waveforms not to overlap each other, two targets need to be separated by $c\tau/2$. This range is usually referred to as range resolution i.e. $R_{res} = c\tau/2$. Using minimum separation requirement, if the target has a response with duration T and that the eigenwaveform has a time duration $\tau = T$, the falling edge of one target return has to be separated by $c\tau/2 = cT/2$ from the leading edge of another target return as illustrated in Fig. 1b. Thus, the range and temporal resolution for an extended target illuminated by an eigenwaveform are

$$R_{res} = c\tau/2 + cT = \frac{3}{2}cT. \tag{1}$$

$$T_{res} = 1.5T, \tag{2}$$

i.e. the separation has to be a least by $1.5T$. In top panel of Fig. 2, we illustrate two extended targets (having the same complex-valued impulse response) to have a time separation of $2T$. Using the corresponding eigenwaveform, notice the magnitude returns are not back-to-back. In bottom panel, we use $1.5T$ separation. Notice that the target returns are back-to-back. Thus, $1.5T$ is the minimum time separation such that the targets do not overlap in range.

III. AMBIGUITY FUNCTION

For transmit signal $x(t)$ and target response $h(t)$, let $s(t)$ be the convolution of transmit signal and target response. Assume a matched filter in the receiver for signal detection. The output of matched filter is

$$\chi(\tau) = \int_{-\infty}^{\infty} s(t)s^*(t-\tau)e^{j2\pi f_d}dt, \tag{3}$$

where f_d is the Doppler shift of a certain return and $s(t)$ is the convolution result of transmit signal and target response. The ambiguity function (or error function) is defined in [3] as

$$|\chi(\tau)|^2 = \left|\int_{-\infty}^{\infty} s(t)s^*(t-\tau)e^{j2\pi f_d}dt\right|^2. \tag{4}$$

But for an extended target, the return is given by $s(t) = h(t) * x(t)$, thus the ambiguity function is modified into

$$|\chi(\tau)|^2 = \left|\int_{-\infty}^{\infty} h(t) * x(t)h^*(t-\tau)x^*(t-\tau)e^{j2\pi f_d}dt\right|^2. \tag{5}$$

Clearly, the ambiguity function is a function of both transmit waveform and target response. Two properties of the ambiguity functions are usually of interest.

The first property is the peak of ambiguity function. If the energy of signal is $E_x = \int_{-\infty}^{\infty} |x(t)|^2 dt$, the maximum value of ambiguity function will be found at $\tau = 0$ and $f_d = 0$, and the peak value of the ambiguity function is easily given by

$$\begin{aligned}&|\chi(\tau; f_d)|^2_{\max} = |\chi(0;0)|^2 \\ &= \left|\int_{-\infty}^{\infty} s(t)s^*(t)dt\right|^2 = |E_s|^2 = E_s^2\end{aligned} \tag{6}$$

For a unit-amplitude point target, it is clear that $E_s = E_x$. For the extended target, the eigenfunction $q_{\max}(t)$ corresponding to the maximum eigenvector $\lambda_{\max}$ is used as the transmit waveform, *aka* eigenwaveform [4] [5]. Therefore

$$x(t) = q_{\max}(t),$$

where

$$\lambda_{\max} q_{\max}(t) = \int_{-\infty}^{\infty} q_{\max}(t)L(\rho-\tau)d\rho d\tau \tag{7}$$

and

$$L(\tau) = \int_{-\infty}^{\infty} |H(f)|^2 e^{j2\pi f\tau} df. \tag{8}$$

As a result, the maximum value of ambiguity function is

$$\begin{aligned}&|\chi(\tau; f_d)|^2_{\max} = |\chi(0;0)|^2 \\ &= \left|\int_{-\infty}^{\infty} [q_{\max}(t) * h(t)][q_{\max}(t) * h(t)]^* dt\right|^2 \\ &= \left|\int_{-\infty}^{\infty} Q(f)Q^*(f)|H(f)|^2 df\right|^2.\end{aligned}$$

By substituting (7) and (8) into (4), E_s become

$$\begin{aligned}E_s &= \int_{-\infty}^{\infty} q_{\max}(\tau)\lambda_{\max} {q_{\max}}^*(\tau)d\tau \\ &= \lambda_{\max} E_{q_{\max}}.\end{aligned} \tag{9}$$

Thus, the maximum value of ambiguity function is

$$|\chi(0;0)|^2 = \lambda^2_{\max} E^2_{q_{\max}} = \lambda^2_{\max} E^2_x.$$

Compared to traditional ambiguity function whose peak is E_x^2, the peak value for an eigenwaveform illuminating an extended target is amplified by $\lambda^2_{\max}$.

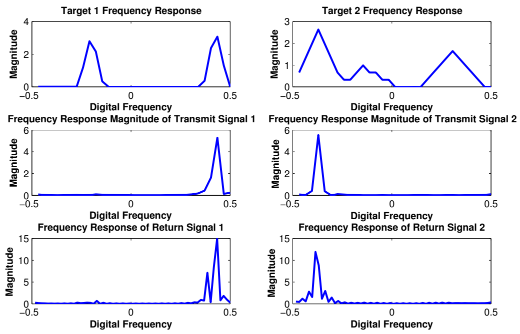

Fig. 3. Top panel: Target magnitude in frequency response. Middle panel: Frequency response magnitude of transmit signal. Bottom panel: Frequency response of return signal

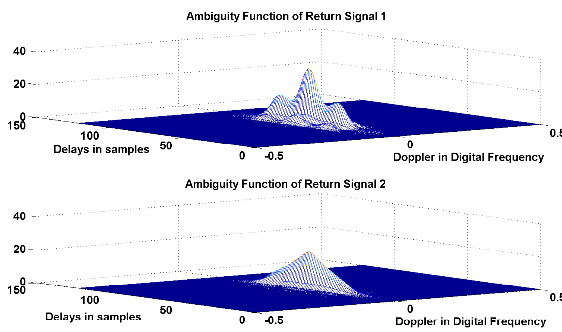

Fig. 4. Top panel: Ambiguity function of sample extended target 1. Bottom panel: Ambiguity function of sample extended target 2.

In practice, due to the advent of arbitrary waveform generators, the waveforms are designed in discrete-time and then converted to continuous time with the use of digital-to-analog converter prior to RF upconversion. Thus, assuming proper sampling rate, we can utilize discrete-time model where $\mathbf{h}$ is target response vector, $\mathbf{x}$ is transmit waveform vector, and $\mathbf{s} = \mathbf{h} * \mathbf{x}$ is the target return. We can form a target convolution matrix given by

$$\mathbf{H} = \begin{bmatrix} \mathbf{h} & 0 & 0 & 0 \\ . & \mathbf{h} & 0 & 0 \\ . & . & .. & .. \\ . & & & \mathbf{h} \end{bmatrix}$$

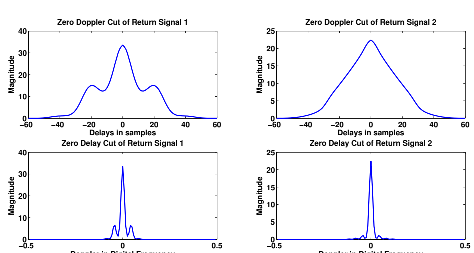

Fig. 5. Top panel: Zero Doppler cut. Bottom panel: Zero delay cut.

and the energy of the return is given by

$$E_s = (\mathbf{Hx})^H \mathbf{Hx} = \mathbf{x}^H \mathbf{H}^H \mathbf{Hx}.$$

Let $\mathbf{R_H} = \mathbf{H}^H \mathbf{H}$ and $\mathbf{x}$ be the eigenvector $\mathbf{q}_{\max}$ corresponding to largest eigenvalue $\lambda_{\max}$ of $\mathbf{R_H}$. Therefore,

$$\begin{aligned} E_s &= \mathbf{x}^H \mathbf{H}^H \mathbf{Hx} = \mathbf{x}^H \mathbf{R_H} \mathbf{x} \\ &= \mathbf{q}_{\max}{}^H \lambda_{\max} \mathbf{q}_{\max} = \lambda_{\max} E_x. \end{aligned} \tag{10}$$

So,

$$|\chi(\tau; f_d)|^2_{\max} = |\chi(0;0)|^2 = E_s^2 = \lambda^2_{\max} E_x^2.$$

In other words, the peak of the ambiguity function which is traditionally the squared-energy of the transmit signal is potentially larger for an extended target illuminated by an eigenwaveform. That is, the peak is amplified by the squared-eigenvalue, which results in tremendous gain especially if the dominant eigenvalue is large!

We consider two complex-valued extended targets illustrated in Fig. 3. In the top panel, the magnitude of the target frequency responses (normalized) are shown. Both targets are of unit energy. The magnitude of the frequency responses of eigenwaveform are shown in the middle panel. The bottom panel shows the magnitude of the frequency responses of the return signals. The ambiguity functions are shown in Fig. 4. Notice that these ambiguity functions are different from ambiguity function of traditional radar waveforms such as pulsed waveform. For an unit-energy transmit waveform illuminating a unit-amplitude point target, the traditional peak of the ambiguity function is 1. In this example, we consider the unit-energy target 1 in Fig. 3 and use an unit energy transmit signal, i.e. the eigenwaveform has an unit energy. Notice that the peak of the ambiguity function for the extended target 1 in Fig. 4 (illuminated by a unit-energy eigenwaveform) is $\lambda^2_{\max}$. In this example, since the maximum eigenvalue is 5.79, the peak is 33.53 which is a substantial gain of 15.3 dB! In Fig. 5, we illustrate the zero-Doppler cut and zero-delay cuts for the sample extended targets.

Another property that is of interest is the total volume under the ambiguity function is constant and in our case is given by

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |\chi(\tau; f_d)|^2 d\tau df_d = E_s^2 = \lambda^2_{\max} E_x^2.$$

Range-Doppler map (RDM) is one of the ambiguity function applications that significantly demonstrates the benefit of using extended target and eigenwaveform. A sequence of transmit waveforms (pulses) are sent, received and sampled, matched filtered, and a measurement matrix is formed. After FFT, the magnitude is taken to form the range-Doppler map [2]. In Fig. 6, we illustrate the RDM due to the presence of a target 1 (in Fig. 4) with the use of a series of wideband pulses in the presence of AWGN. The target is somewhat discernable after matched filtering. In Fig. 7, using the same target and same amount of noise, we use a series of eigenwaveforms. After the matched filter, notice the substantial peak due to the use of eigenwaveform pulses.

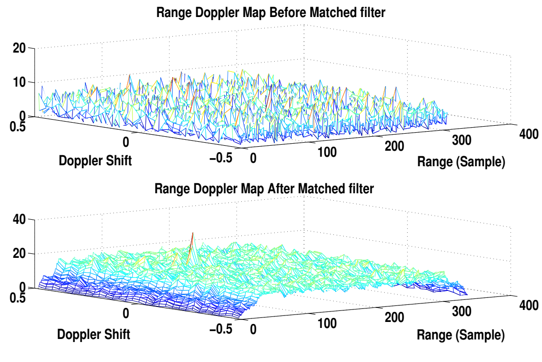

Fig. 6. Range Doppler map for point target scenario, Top panel: Before matched filter. Bottom panel: After matched filter.

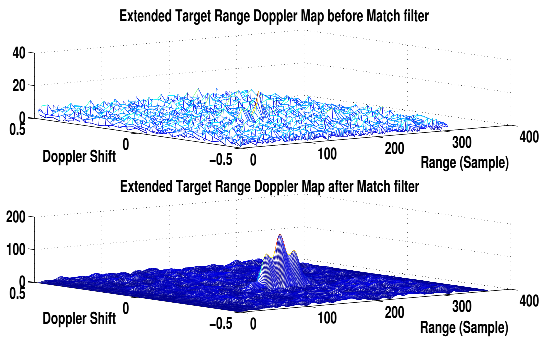

Fig. 7. Extended target scenario, Top panel: Before matched filter. Bottom panel: After matched filter.

IV. Probability of Detection

Here we consider the detection performance for the eigenwaveform design. The detection hypotheses are

$$H_0 : \mathbf{y} = \mathbf{w}$$
$$H_1 : \mathbf{y} = \mathbf{s} + \mathbf{w} = \mathbf{H}\mathbf{x} + \mathbf{w}.$$

It can be shown that the decision statistic for fixed threshold γ is

$$T(\mathbf{y}) = \mathrm{Re}\{\mathbf{y}^H \mathbf{H}\mathbf{x}\}.$$

Under either hypothesis, $T(\mathbf{y})$ is Gaussian since it is a linear combination of Gaussian random variables. And the expected values under two hypotheses are

$$E(T; H_0) = E(\mathbf{w}^H \mathbf{s}) = 0$$
$$E(T; H_1) = E(\mathbf{s}^H \mathbf{s} + \mathbf{w}^H \mathbf{s}) = E_s = \lambda_{\max} E_x.$$

And the variance under the null hypothesis is

$$\mathrm{var}(T; H_0) = \mathrm{var}(T; H_1) = \mathrm{var}(\mathbf{w})\mathbf{s}^H\mathbf{s} = \frac{\sigma^2 E_s}{2} = \frac{\sigma^2 \lambda_{\max} E_x}{2}$$

It can be shown that the probability of detection is

$$P_D = \Pr(T \geq \gamma^{'}; H_1)$$
$$= Q\Big(\frac{\gamma^{'} - E_s}{\sqrt{\frac{\sigma^2 E_s}{2}}}\Big) = Q\Big(Q^{-1}(P_{FA}) - \sqrt{\frac{2\lambda_{\max} E_x}{\sigma^2}}\Big).$$

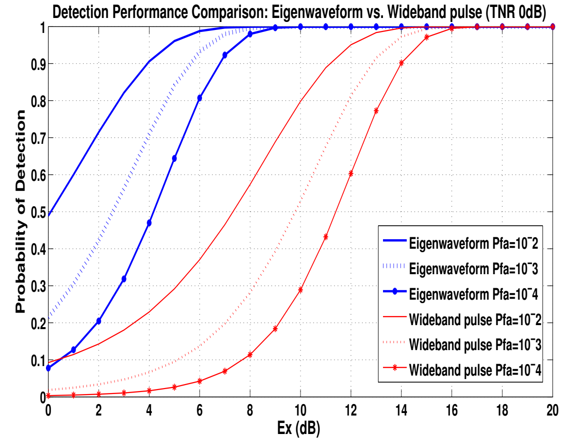

Fig. 8. Probability of detection for eigenwaveform vs. wideband pulse

Notice that the probability of detection is a function of the Q-function. In the case of traditional waveform design for point targets or wideband pulse for extended target, the probability is purely a function of $E_s = E_x \times E_h$ rather than $\lambda_{\max} E_x$. In either case, the detection performance is

$$P_D = \Pr(T \geq \gamma^{'}; H_1)$$
$$= Q\Big(\gamma^{'} - E_s\sqrt{\frac{\sigma^2 E_s}{2}}\Big) = Q\Big(Q^{-1}(P_{FA}) - \sqrt{\frac{2E_x E_h}{\sigma^2}}\Big)$$

In eigenwaveform design, E_x is amplified by the eigenvalue of matrix $\mathbf{H}$ which can easily be larger than E_h and as such has better performance than point target or wideband pulse cases. To show the performance comparison using a pulsed waveform illuminating a sample extended target as opposed to eigenwaveform, we plot the detection probability of both waveforms in Fig. 8. Notice the superior performance of the eigenwaveform.

For extended targets, what's interesting is that different targets will have different maximal eigenvalues. Since extended targets will yield different eigenwaveforms, detection probability depends from target to target (even if the energy of their responses are the same).

V. Summary

In this paper, we investigated the range resolution of a radar employing an eigenwaveform illuminating an extended target. To that end, we derived the ambiguity function for this case. The peak of the ambiguity function is the transmit energy (squared) multiplied by the maximum eigenvalue (squared) as opposed to traditional waveforms where the peak is just the transmit energy (squared). Also, we look at the detection probability improvement for eigenwaveforms matched to extended targets. The detection performance is clearly better than just employing a wideband pulsed waveform.

References

[1] S. M. KAY, *Fundamentals of Statistical Signal Processing volume* , Prentice Hall, Upper Saddle River, NJ, 1993

[2] Nadav Levanon and Eli Mozeson, *Radar Signals*, John Wiley and Sons, NJ, 2004

[3] B. R. Mahafza, *Radar Signal Analysis and Processing using MATLAB*, CRC Press, Boca Raton, FL, 2009

[4] M. R. Bell, "information theory and radar waveform design," *IEEE. Trans. information Theory.*, vol. 39, no. 5, pp. 1578-1597, Sep. 1993.

[5] Ric. A. Romero, J. BAE, and N. A. Goodman , "Theory and Apllication of SNR and Mutual information Matched illumination Waveforms, *IEEE Trans. Aerosapce and Electronic System.*, vol. 47, no. 2, pp. 912-927, Apr 2011.